Understanding n-dimensional rotations: the derivation of the Hypercross Product.

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I. Abstract

The cross-product operation on two vectors is a well-known operation that is applicable to physics. The operation is a vector product which yields a third vector that is orthogonal to the original two vectors and has a magnitude of the product of the magnitudes of the two individual vectors multiplied by the sine of the angle between the two vectors. It is an extraordinarily useful formula in physics which allows us to describe torque and angular momentum essentially giving us understanding of how 3-dimensional rotations work, with the well-known formula: torque(scalar) = $|r||F|\sin\theta$ or torque(vector)= r × F. It is also used in the computation of the curl of vector fields. However, while the cross product is useful for 3-dimensions, it cannot be applied to 4-dimensional or any higher-dimensional rotations. The task of this study was to find a formula which like the cross product can be used to describe rotations in 4-dimensional Euclidean space, as well as the generalization of this idea to any n-dimensional Euclidean space. A formula was found with properties similar to the cross product, which will be called the 4-hypercross product. This formula can be extended to n-dimensional Euclidean spaces where n is any natural number greater than or equal to 2, creating the n-hypercross product.

II. Introduction

While it may seem mysterious at first, the reason why there is no cross product in 4-dimensional spaces, in other words, why the cross product cannot be applied to 4-dimensions, is because of the fact that rotations occur around an axis in 3 dimensions, and around a plane in 4 dimensions. In fact, rotations occur around the $n-2$ dimensional hyperplane for any dimension n (point for 2 dimensions, line for 3 dimensions, plane for 4 dimensions, 3-hyperplane for 5 dimensions, etc.). So, the cross product which yields a vector must be modified to yield the 2 and higher-dimensional analogue, a matrix. A 3×1 vector is yielded for the cross product since it is a line that exists in a 3-dimensional space. Since we are now searching for a plane that exists in a

4-dimensional space, we seek a 4×2 matrix for the 4-hypercross product. The matrix that we are seeking should be of the form:

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}$$

We will use a new symbol to define this hypercross product between two vectors, $\vec{a}$ and $\vec{b}$, (which are 4×1 vectors) which will be $\vec{a}\,\mathbf{X}\,\vec{b}$, so

$\vec{a}\,\mathbf{X}\,\vec{b} =$

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}$$

III. Linear Algebra

In order to find a similar operation to the cross product in 4 dimensions, we need to find the orthogonal complement of the two vectors in $\mathbb{R}^4$ that represent the rotation (the arm and the motion vector, or the arm and the force vector). In order to do this, we use the concepts of image and kernel from linear algebra.

In mathematics, the image of a linear transformation or a matrix transformation consists of all the values the transformation takes in its target space. It is essentially the span of the concatenated vectors making up the matrix, where the span of all possible values for a location that the sum of scalar multiples of the vectors can take up in space. The kernel of a linear transformation on the other hand is related to the original space of the linear transformation, not the target space. It is defined as all subsets of the domain of $\mathbb{R}^m$ in a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ which map to the 0 vector of $\mathbb{R}^n$. In other words, the kernel of a matrix A is the solution of the linear system $A\vec{x} = \vec{0}$. To solve for the kernel of a matrix we solve that linear system.

We need a few more concepts in order to start deriving this 4-hypercross product. An operation that can be performed on any two $n \times 1$ vectors is the dot product. The dot product is defined as:

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i \,,$$

The dot product gives a scalar which is equal to:

$$|\vec{a}||\vec{b}| \cos\theta$$

Where θ is the angle between the two vectors. Thus, the dot product of two perpendicular vectors is 0. The matrix product of two matrices, $n \times p$ and $p \times m$ is a product which yields a third matrix that is $n \times m$. If A and B are matrices of the aforementioned properties with arbitrary entries, $C = AB$ in the manner such that:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & \vdots \\ \vdots & \dots & \dots & \\ a_{n1} & \dots & \dots & a_{np} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & \vdots \\ \vdots & \dots & \dots & \\ b_{p1} & \dots & \dots & b_{pm} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & \vdots \\ \vdots & \dots & \dots & \\ c_{n1} & \dots & \dots & c_{nm} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

The transpose of a matrix A^T is a matrix whose ij-th entry is the ji-th entry of A. For example, the transpose of the matrix $A = \begin{bmatrix} 4 & -4 & 8 & -9 \\ 3 & 2 & 13 & 1 \end{bmatrix}$ is $A^T = \begin{bmatrix} 4 & 3 \\ -4 & 2 \\ 8 & 13 \\ -9 & 1 \end{bmatrix}$.

Applying the concept of the transpose to a vector, we find that the transpose of a column vector is just a row vector. This is illustrated here:

$$\vec{a} = \begin{bmatrix} 4 \\ 8 \\ 7 \end{bmatrix} \quad \overrightarrow{a^T} = \begin{bmatrix} 4 & 8 & 7 \end{bmatrix}$$

Using this fact, we find that for two column vectors in $\mathbb{R}^n$, their dot product is just the matrix product of the transpose of the first vector and the second vector.

$$\vec{v} \cdot \vec{w} = \overrightarrow{v^T}\, \vec{w}$$

The crucial link that will lead us to derive the hypercross-product is the notion of the orthogonal complement of a subspace of $\mathbb{R}^n$. This is defined as the region of $\mathbb{R}^n$ that contains all vectors that are perpendicular to the vectors in the image of a matrix A. Suppose $V = im(A)$ is a subspace of $\mathbb{R}^n$.

$$A = [\overrightarrow{v_1} \quad \overrightarrow{v_2} \ \overrightarrow{v_3} \quad \dots. \ \dots. \quad \overrightarrow{v_m}]$$

The orthogonal complement $V^{\perp}$ must satisfy the following property:

$$V^{\perp} = all\, \vec{x}\ in\ \mathbb{R}^n \text{ such that } \overrightarrow{v_i} \cdot \vec{x} = 0 \quad for\ all\ i = 1,2,3..m$$

$$V^{\perp} = all\, \vec{x}\ in\ \mathbb{R}^n \text{ such that } \overrightarrow{v_i^T}\, \vec{x} = 0 \quad for\ all\ i = 1,2,3..m$$

So we can say that $V^{\perp} = im(A)^{\perp}$ is just the kernel of the matrix:

$$A^T = \begin{bmatrix} \overrightarrow{v_1^T} \\ \overrightarrow{v_2^T} \\ .. \\ .. \\ .. \\ \overrightarrow{v_m^T} \end{bmatrix}$$

This proves the theorem that given a matrix A

$$im(A)^{\perp} = \ker(A^T)$$

Where ker() denotes the kernel of a matrix. Given that we are searching for the orthogonal complement of the two rotation vectors (the arm and the motion vector), to find the hypercross product we find the kernel of the transpose of the matrix that is the concatenation of these two vectors. This will give us a matrix that spans the orthogonal complement of the rotation vectors, giving us the hyperplane of rotation. So, the hypercross product is essentially the kernel of the transpose of a general 4×2 matrix (note that this method can be used to generate the cross product of two 3×1 vectors as well).

IV. Derivation and Verification

We start by defining the two vectors that are being hypercrossed:

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

We define the matrix K which is the concatenation of these two vectors in $\mathbb{R}^4$, and its transpose.

$$K = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{bmatrix} \qquad K^T = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

To find ker (K^T) we solve the linear system

$$K^T\vec{x} = \vec{0}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First divide the first row by a_1 to get:

$$\left[\begin{array}{cccc|c} 1 & \frac{a_2}{a_1} & \frac{a_3}{a_1} & \frac{a_4}{a_1} & 0 \\ b_1 & b_2 & b_3 & b_4 & 0 \end{array}\right]$$

Subtract b_1 times the first row from the second row to get:

$$\left[\begin{array}{cccc|c} 1 & \frac{a_2}{a_1} & \frac{a_3}{a_1} & \frac{a_4}{a_1} & 0 \\ 0 & b_2 - \frac{b_1 a_2}{a_1} & b_3 - \frac{b_1 a_3}{a_1} & b_4 - \frac{b_1 a_4}{a_1} & 0 \end{array}\right]$$

Divide the second row by $b_2 - \frac{b_1 a_2}{a_1}$:

$$\left[\begin{array}{cccc|c} 1 & \frac{a_2}{a_1} & \frac{a_3}{a_1} & \frac{a_4}{a_1} & 0 \\ 0 & 1 & \frac{b_3 - \frac{b_1 a_3}{a_1}}{b_2 - \frac{b_1 a_2}{a_1}} & \frac{b_4 - \frac{b_1 a_4}{a_1}}{b_2 - \frac{b_1 a_2}{a_1}} & 0 \end{array}\right]$$

We now subtract $\frac{a_2}{a_1}$ times the second row from the first row:

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{a_3}{a_1} - \frac{a_2}{a_1}\left(\frac{b_3 - \frac{b_1 a_3}{a_1}}{b_2 - \frac{b_1 a_2}{a_1}}\right) & \frac{a_4}{a_1} - \frac{a_2}{a_1}\left(\frac{b_4 - \frac{b_1 a_4}{a_1}}{b_2 - \frac{b_1 a_2}{a_1}}\right) & 0 \\ 0 & 1 & \frac{b_3 - \frac{b_1 a_3}{a_1}}{b_2 - \frac{b_1 a_2}{a_1}} & \frac{b_4 - \frac{b_1 a_4}{a_1}}{b_2 - \frac{b_1 a_2}{a_1}} & 0 \end{array}\right]$$

This can be simplified to:

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{a_3}{a_1} - \left(\frac{a_2 b_3 - \frac{a_2 b_1 a_3}{a_1}}{a_1 b_2 - a_2 b_1}\right) & \frac{a_4}{a_1} - \left(\frac{a_2 b_4 - \frac{a_2 b_1 a_4}{a_1}}{a_1 b_2 - a_2 b_1}\right) & 0 \\ 0 & 1 & \frac{a_1 b_3 - a_3 b_1}{a_1 b_2 - a_2 b_1} & \frac{a_1 b_4 - a_4 b_1}{a_1 b_2 - a_2 b_1} & 0 \end{array}\right]$$

We find our solution to be:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} \left(\frac{a_2 b_3 - \frac{a_2 b_1 a_3}{a_1}}{a_1 b_2 - a_2 b_1}\right) - \frac{a_3}{a_1} \\ \frac{a_3 b_1 - a_1 b_3}{a_1 b_2 - a_2 b_1} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \left(\frac{a_2 b_4 - \frac{a_2 b_1 a_4}{a_1}}{a_1 b_2 - a_2 b_1}\right) - \frac{a_4}{a_1} \\ \frac{a_4 b_1 - a_1 b_4}{a_1 b_2 - a_2 b_1} \\ 0 \\ 1 \end{bmatrix}$$

The solution space is the span of these two vectors. We can concatenate them to get a matrix:

$$\begin{bmatrix} \left(\frac{a_2 b_3 - \frac{a_2 b_1 a_3}{a_1}}{a_1 b_2 - a_2 b_1}\right) - \frac{a_3}{a_1} & \left(\frac{a_2 b_4 - \frac{a_2 b_1 a_4}{a_1}}{a_1 b_2 - a_2 b_1}\right) - \frac{a_4}{a_1} \\ \frac{a_3 b_1 - a_1 b_3}{a_1 b_2 - a_2 b_1} & \frac{a_4 b_1 - a_1 b_4}{a_1 b_2 - a_2 b_1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Scale the matrix by a factor of $a_1 b_2 - a_2 b_1$ to get:

$$\begin{bmatrix} a_2 b_3 - a_3 b_2 & a_2 b_4 - a_4 b_2 \\ a_3 b_1 - a_1 b_3 & a_4 b_1 - a_1 b_4 \\ a_1 b_2 - a_2 b_1 & 0 \\ 0 & a_1 b_2 - a_2 b_1 \end{bmatrix}$$

This is the orthogonal complement of the vectors $\vec{a}$ and $\vec{b}$. We define the hypercross product of vectors $\vec{a}$ and $\vec{b}$

$$\vec{a} \mathbin{\text{⧗}} \vec{b} = \begin{bmatrix} a_2b_3 - a_3b_2 & a_2b_4 - a_4b_2 \\ a_3b_1 - a_1b_3 & a_4b_1 - a_1b_4 \\ a_1b_2 - a_2b_1 & 0 \\ 0 & a_1b_2 - a_2b_1 \end{bmatrix}$$

We have found the orthogonal complement, but we still do not know the area spanned by the parallelogram that is formed by these two concatenated vectors. To find the area spanned by the vectors of a non-square matrix, we use the formula for a parallelepiped of order m spanned by m vectors in $\mathbb{R}^n$. The m-volume is:

$$\sqrt{\det(A^T A)}$$

We must find the matrix product of the 4-hypercross product matrix and is transpose, find the determinant of this matrix, and then find the square root of this determinant. First, perform matrix multiplication on the matrices A^T and $A = \vec{a} \mathbin{\text{⧗}} \vec{b}$:

$$A^T A$$

$$= \begin{bmatrix} a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 & 0 \\ a_2b_4 - a_4b_2 & a_4b_1 - a_1b_4 & 0 & a_1b_2 - a_2b_1 \end{bmatrix} \begin{bmatrix} a_2b_3 - a_3b_2 & a_2b_4 - a_4b_2 \\ a_3b_1 - a_1b_3 & a_4b_1 - a_1b_4 \\ a_1b_2 - a_2b_1 & 0 \\ 0 & a_1b_2 - a_2b_1 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 & (a_2b_4 - a_4b_2)(a_2b_3 - a_3b_2) + (a_4b_1 - a_1b_4)(a_3b_1 - a_1b_3) \\ (a_2b_4 - a_4b_2)(a_2b_3 - a_3b_2) + (a_4b_1 - a_1b_4)(a_3b_1 - a_1b_3) & (a_2b_4 - a_4b_2)^2 + (a_4b_1 - a_1b_4)^2 + (a_1b_2 - a_2b_1)^2 \end{bmatrix}$$

The determinant of this matrix is then computed:

$$\det(A^T A) =$$

$$((a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2)((a_2b_4 - a_4b_2)^2 + (a_4b_1 - a_1b_4)^2 + (a_1b_2 - a_2b_1)^2)$$
$$- ((a_2b_4 - a_4b_2)(a_2b_3 - a_3b_2) + (a_4b_1 - a_1b_4)(a_3b_1 - a_1b_3))^2$$

We do lots of algebra and expansion to simplify this expression to:

$$\det\left([\vec{a}\,\mathbf{X}\,\vec{b}\,]^T[\vec{a}\,\mathbf{X}\,\vec{b}]\right)$$
$$= (a_1b_2 - a_2b_1)^2((a_1b_2 - a_2b_1)^2 + (a_3b_1 - a_1b_3)^2 + (a_2b_3 - a_3b_2)^2 + (a_4b_1 - a_1b_4)^2$$
$$+ (a_2b_4 - a_4b_2)^2 + (a_4b_3 - a_3b_4)^2)$$

Using the Pythagorean theorem and knowledge of the dot product, we can find $|\vec{a}||\vec{b}|\sin\theta$:

$$|\vec{a}|^2|\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2\sin^2\theta + |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$
$$|\vec{a}|^2|\vec{b}|^2\sin^2\theta = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$
$$|\vec{a}|^2|\vec{b}|^2\sin^2\theta = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$$

So, we square the magnitudes of the vectors and subtract the square of their dot product to get:

$$|\vec{a}|^2|\vec{b}|^2\sin^2\theta =$$
$$({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2)({b_1}^2 + {b_2}^2 + {b_3}^2 + {b_4}^2) - (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4)^2$$
$$= (a_1b_2 - a_2b_1)^2 + (a_3b_1 - a_1b_3)^2 + (a_2b_3 - a_3b_2)^2 + (a_4b_1 - a_1b_4)^2 + (a_2b_4 - a_4b_2)^2 + (a_4b_3 - a_3b_4)^2$$

Noticing the similarities between the determinant and this expression, we have found the equation:

$$|\vec{a}|^2|\vec{b}|^2\sin^2\theta = \frac{\det\left([\vec{a}\,\mathbf{X}\,\vec{b}\,]^T[\vec{a}\,\mathbf{X}\,\vec{b}]\right)}{(a_1b_2 - a_2b_1)^2}$$

Take the square root on both sides to get:

$$|\vec{a}||\vec{b}|\sin\theta = \frac{\sqrt{\det\left([\vec{a}\,\mathbf{X}\,\vec{b}\,]^T[\vec{a}\,\mathbf{X}\,\vec{b}]\right)}}{a_1b_2 - a_2b_1}$$

So, this matrix that we have found, with just one additional term, is able to provide information about the parallelogram spanned by the vectors being hypercrossed! Recall that $\sqrt{\det(A^TA)}$ is the 2-volume spanned by the vectors that are the concatenated in the hypercross matrix. Compare this to the formula for $|\vec{a}||\vec{b}|\sin\theta$ with two 3x1 vectors,

$$|\vec{a}||\vec{b}|\sin\theta = |\vec{a}\times\vec{b}|$$

We can make a slight modification to both formulas to find this interesting relationship, with D being the dimension (number of rows) of the one column vectors $\vec{a}$ and $\vec{b}$:

D = 3:

$$|\vec{a}||\vec{b}|\sin\theta = |\vec{a}\times\vec{b}| = \frac{|\vec{a}\times\vec{b}|}{(a_1 b_2 - a_2 b_1)^{3-3}} = \frac{|\vec{a}\times\vec{b}|}{(a_1 b_2 - a_2 b_1)^{0}}$$

D=4

$$|\vec{a}||\vec{b}|\sin\theta = \frac{\sqrt{\det\left([\vec{a}\,\text{⧗}\,\vec{b}\,]^T[\vec{a}\,\text{⧗}\,\vec{b}]\right)}}{a_1 b_2 - a_2 b_1} = \frac{\sqrt{\det\left([\vec{a}\,\text{⧗}\,\vec{b}\,]^T[\vec{a}\,\text{⧗}\,\vec{b}]\,\right)}}{(a_1 b_2 - a_2 b_1)^{4-3}}$$
$$= \frac{\sqrt{\det\left([\vec{a}\,\text{⧗}\,\vec{b}\,]^T[\vec{a}\,\text{⧗}\,\vec{b}]\right)}}{(a_1 b_2 - a_2 b_1)^{1}}$$

Add D = 2 and we find that this rule is extended.

$$|\vec{a}||\vec{b}|\sin\theta = a_1 b_2 - a_2 b_1 = \frac{1}{(a_1 b_2 - a_2 b_1)^{2-3}} = \frac{1}{(a_1 b_2 - a_2 b_1)^{-1}}$$

V. Applications and extension to the general n-hypercross product

The 4-hypercross product, with its properties being very similar to the 3-dimensional cross product, can be used to describe rotations in 4 dimensional spaces. We can define the 4-dimensional angular momentum formula to be

$$L = \vec{r}\,\text{⧗}\,\mathrm{m}\vec{v} = \vec{r}\,\text{⧗}\,\vec{p}$$

Whereas the formula for angular force, also known as moment or torque would be:

$$\tau = \vec{r}\,\text{⧗}\,\mathrm{m}\vec{a} = \vec{r}\,\text{⧗}\,\vec{F}$$

These would give the angular momentum and torque matrices which represent a subset of the orthogonal subspace, or plane of rotation. Any linear combination of the two column vectors of this matrix lie on the plane of rotation. For the angular momentum matrix, it has an area of

$g_{4-0}|\vec{r}||\vec{p}|\sin\theta$, where $g_{4-0} = r_1p_2 - r_2p_1$ is an extra quantity used for the calculation known as the 2-volume momentum factor. For the torque matrix, it has an area of $g_{4-1}|\vec{r}||\vec{F}|\sin\theta$, where $g_{4-1} = r_1F_2 - r_2F_1$ is an extra quantity used for the calculation known as the 2-volume force factor. This is in the case that these matrices do not have any columns which are just the zero vector, the columns are linearly independent, as well as the $g_{4-(0,1)}$factor being non-zero.

We can define a general n-hypercross product by using the same method determined to find the 4-hypercross product, finding the kernel of the transpose of a general $n \times (n-2)$ matrix. The n-hypercross product, denoted as

$$\vec{a} \text{ ⧗}_{\mathrm{n}} \vec{b}$$

Can be determined through the same linear algebra algebraic expansions to be equal to an $n \times (n-2)$ matrix of the form:

$$\begin{bmatrix} A \\ B \end{bmatrix}$$

Where A is a $n \times (n-2)$ matrix f the form

$$\begin{bmatrix} a_2b_3 - a_3b_2 & a_2b_4 - a_4b_2 & a_2b_5 - a_5b_2 & a_2b_6 - a_6b_2 & \ldots\ldots. & a_2b_n - a_nb_2 \\ a_3b_1 - a_1b_3 & a_4b_1 - a_1b_4 & a_5b_1 - a_1b_5 & a_6b_1 - a_1b_6 & \ldots\ldots.. & a_nb_1 - a_1b_n \end{bmatrix}$$

And B is

$$(a_1b_2 - a_2b_1)I_{n-2}$$

Where I_{n-2} denotes the identity matrix of order $n-2$. Angular momentum and torque in n-dimensions can be defined through the same manner, with factors g_{n-0} and g_{n-1} , the n-2 volume momentum and force factors, respectively being equal to $(r_1p_2 - r_2p_1)^{n-3}$ and $(r_1F_2 - r_2F_1)^{n-3}$. The general angular momentum and torque matrices for an n-dimensional Euclidean space of dimension 4 or greater are n x n-2 matrices that can be represented by the formulas:

$$L = \vec{r} \text{ ⧗}_{\mathrm{n}} \mathrm{m}\vec{v} = \vec{r} \text{ ⧗}_{\mathrm{n}} \vec{p}$$

$$\tau = \vec{r} \text{ ⧗}_{\mathrm{n}} \mathrm{m}\vec{a} = \vec{r} \text{ ⧗}_{\mathrm{n}} \vec{F}$$

We find that there is a rule for all dimensions in which rotations are possible, that is all $d \geq 2$. The area spanned by the parallelogram formed by the two vectors defining the rotation, which has a scalar value of the magnitude of the torque or angular momentum, follows the rule described in this table:

Dimension	$\lvert\vec{a}\rvert\lvert\vec{b}\rvert \sin\theta$	Rotation hyperplane
2	$\dfrac{1}{(a_1b_2 - a_2b_1)^{-1}}$	Point of rotation
3	$\dfrac{\lvert\vec{a}\times\vec{b}\rvert}{(a_1b_2 - a_2b_1)^{0}}$	Axis of rotation
4	$\dfrac{\sqrt{\det\left([\vec{a} \mathbin{⧗} \vec{b}\,]^T[\vec{a} \mathbin{⧗} \vec{b}]\right)}}{a_1b_2 - a_2b_1}$	Plane of rotation
5	$\dfrac{\sqrt{\det\left([\vec{a} \mathbin{⧗}_5 \vec{b}\,]^T[\vec{a} \mathbin{⧗}_5 \vec{b}]\right)}}{(a_1b_2 - a_2b_1)^{2}}$	3-hyperplane of rotation
6	$\dfrac{\sqrt{\det\left([\vec{a} \mathbin{⧗}_6 \vec{b}\,]^T[\vec{a} \mathbin{⧗}_6 \vec{b}]\right)}}{(a_1b_2 - a_2b_1)^{3}}$	4-hyperplane of rotation
n (all n greater than or equal to 4)	$\dfrac{\sqrt{\det\left([\vec{a} \mathbin{⧗}_n \vec{b}\,]^T[\vec{a} \mathbin{⧗}_n \vec{b}]\right)}}{(a_1b_2 - a_2b_1)^{n-3}}$	(n-2) hyperplane of rotation

So, there is a regular relationship between the volume of the hyperplane of rotation produced by the cross product or n-hypercross product operation, except for the case in 2 dimensions where the volume of a point is hard to define.

The cross product can be applied to the notion of curl which is an important applied concept in physics. The curl of a vector field is defined as the cross product of the del operator and the vector field. It is defined as such:

$$curl\,\vec{F} = \nabla \times \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{bmatrix}$$

Using the curl, we can show that for conservative forces, that is, forces which are the gradient of some sort of a potential (energy), the work done from point a to point b is the same regardless of the path, as well as the fact that the total work done along a closed loop is zero, in 2 and 3-dimensional spaces. This can be extended to n-dimensional spaces using the n-curl which is defined in terms of the n-hypercross product. In order to show this, we must prove a statement about the curl of a gradient field, as well as a theorem known as Stokes' theorem.

For any twice-continuously differentiable function f, the curl of the gradient of f is the zero vector.

$$curl\,\nabla f = \nabla \times \nabla f = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \end{bmatrix} = \vec{0}$$

This is due to the equality of mixed partial derivatives for twice-continuously differentiable functions.

We can hypercross the del operator with a vector field in $\mathbb{R}^4$ to obtain a quantity known as the 4-curl (has some physical similarities with ordinary curl). This gives us the following result:

$$curl_4\,\vec{F} = \nabla \mathbb{X}\,\vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial w} \end{bmatrix} \mathbb{X} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_4}{\partial y} - \frac{\partial F_2}{\partial w} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_1}{\partial w} - \frac{\partial F_4}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} & 0 \\ 0 & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{bmatrix}$$

It has some properties of the ordinary curl. If we take the "4-curl" of a gradient field in 4-d space, the first column vector of this matrix is clearly $\vec{0}$ since it is the same expression for all non-trivial terms as the ordinary curl. If we check the second column vector, we get the same result:

$$curl_4\,\nabla f = \nabla \mathbb{X}\,\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial w} \end{bmatrix} \mathbb{X} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \\ \frac{\partial f}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial y \partial w} - \frac{\partial^2 f}{\partial w \partial y} \\ \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial w \partial x} - \frac{\partial^2 f}{\partial x \partial w} \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} & 0 \\ 0 & \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \end{bmatrix}$$

So, the 4-curl of any gradient with the specified properties is a 4×2 zero matrix.

By applying Stokes' theorem, we can prove that, in $\mathbb{R}^2$ or $\mathbb{R}^3$, the work done by a conservative vector field over a closed loop is zero, and that the work done by the vector field from point a to point b is the same regardless of the path taken. Stokes' theorem relates the integral of the curl of a vector field over a surface to the line integral of the vector field over the boundary of the surface. For some background, it is proven here. The line integral of a force field over a curve is the total work done. It is defined as such for a curve $\vec{c}(t) = (x(t), y(t), z(t))$ and a vector field $\vec{F}$:

$$\int \vec{F} \cdot d\vec{s} = \int_a^b \overrightarrow{F(\vec{c}(t))} \cdot \frac{d}{dt}\vec{c}(t)\,\mathrm{dt}$$

Parametrized surfaces in $\mathbb{R}^3$ are functions Φ from $\mathbb{R}^2$ to $\mathbb{R}^3$ which map a domain D in $\mathbb{R}^2$ to create an image $S = \Phi(\mathrm{D})$. We can write this function as:

$$\Phi(\mathrm{u}, \mathrm{v}) = (\mathrm{x}(\mathrm{u}, \mathrm{v}), \mathrm{y}(\mathrm{u}, \mathrm{v}), \mathrm{z}(\mathrm{u}, \mathrm{v}))$$

The boundary of the curve is defined as a function:

$$\vec{p}(t) = (\mathrm{x}(\mathrm{u}(\mathrm{t}), \mathrm{v}(\mathrm{t})), \mathrm{y}(\mathrm{u}(\mathrm{t}), \mathrm{v}(\mathrm{t})), \mathrm{z}(\mathrm{u}(\mathrm{t}), \mathrm{v}(\mathrm{t})))$$

We define the tangent vectors to the curves on the surface as $\overrightarrow{T_u}$ and $\overrightarrow{T_v}$. These are the vectors:

$$\overrightarrow{T_u} = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{bmatrix} \text{ and } \overrightarrow{T_v} = \begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{bmatrix}$$

Taking the cross product of these two vectors gives a third vector that is normal to the surface, $\overrightarrow{T_u} \times \overrightarrow{T_v}$. The surface integral of a vector field defined on the surface S can be defined using this normal vector. For a vector field $\vec{F}$ in $\mathbb{R}^3$, the surface integral is defined as such:

$$\iint \vec{F} \cdot \mathrm{d}\vec{S} = \iint_D \vec{F} \cdot \left(\overrightarrow{T_u} \times \overrightarrow{T_v}\right) dudv$$

In Stokes' theorem, we consider the case of the vector field being the curl of some vector field. The vector surface integral of a C^1 vector field is:

$$\iint (\nabla \times \vec{F}) \cdot \left(\overrightarrow{T_u} \times \overrightarrow{T_v}\right) dudv$$

The boundary of this surface can be described as a curve. We computed the vector $\overrightarrow{T_u} \times \overrightarrow{T_v}$ which is equal to:

$$\overrightarrow{T_u} \times \overrightarrow{T_v} = \begin{bmatrix} \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u} \\ \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u} \\ \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \end{bmatrix}$$

The surface vector surface integral is defined as:

$$\iint \left(\nabla \times \vec{F}\right) \cdot \mathrm{d}\vec{S}$$
$$= \iint_D \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v})dA$$

We now can relate this to the line-integral over the boundary of the surface.

$$\int \vec{F} \cdot d\vec{s} = \int_a^b F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \mathrm{dt}$$

Then, we can rewrite the derivatives as:

$$\frac{dx}{dt} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt} \ , \quad \frac{dy}{dt} = \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt} \ , \quad \frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}$$

So, the line integral is rewritten as:

$$\int (F_1 \frac{\partial x}{\partial u} + F_2 \frac{\partial y}{\partial u} + F_3 \frac{\partial z}{\partial u}) \frac{du}{dt} + (F_1 \frac{\partial x}{\partial v} + F_2 \frac{\partial y}{\partial v} + F_3 \frac{\partial z}{\partial v}) \frac{dv}{dt} \mathrm{dt}$$

Using Green's theorem, we can rewrite this as:

$$\iint \frac{\partial (F_1 \frac{\partial x}{\partial v} + F_2 \frac{\partial y}{\partial v} + F_3 \frac{\partial z}{\partial v})}{\partial u} - \frac{\partial (F_1 \frac{\partial x}{\partial u} + F_2 \frac{\partial y}{\partial u} + F_3 \frac{\partial z}{\partial u})}{\partial v} dA$$

This can be simplified to be rewritten as:

$$\iint \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}) dA$$

Now lets try and use the hypercross product and its associated "n-curl" to write an identity that works in other dimensions since the cross product cannot be applied to a vector field in $\mathbb{R}^4$ or greater. Now the surface is parametrized as:

$$\Phi(\mathrm{u}, \mathrm{v}) = (\mathrm{x}(\mathrm{u}, \mathrm{v}), \mathrm{y}(\mathrm{u}, \mathrm{v}), \mathrm{z}(\mathrm{u}, \mathrm{v}), \mathrm{w}(\mathrm{u}, \mathrm{v}))$$

And the corresponding boundary is:

$$\vec{p}(t) = (\mathrm{x}(\mathrm{u}(\mathrm{t}), \mathrm{v}(\mathrm{t})), \mathrm{y}(\mathrm{u}(\mathrm{t}), \mathrm{v}(\mathrm{t})), \mathrm{z}(\mathrm{u}(\mathrm{t}), \mathrm{v}(\mathrm{t})), \mathrm{w}(\mathrm{u}(\mathrm{t}), \mathrm{v}(\mathrm{t})))$$

The unit tangent vectors to the surface are:

$$\overrightarrow{T_u} = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \\ \frac{\partial w}{\partial u} \end{bmatrix} \text{ and } \overrightarrow{T_v} = \begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \\ \frac{\partial w}{\partial v} \end{bmatrix}$$

$$\overrightarrow{T_u} \times \overrightarrow{T_v} = \begin{bmatrix} \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u} & \frac{\partial y}{\partial u}\frac{\partial w}{\partial v} - \frac{\partial w}{\partial u}\frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u} & \frac{\partial w}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial w}{\partial v}\frac{\partial x}{\partial u} \\ \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} & 0 \\ 0 & \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \end{bmatrix}$$

Now we can take the line integral of the vector field along the boundary.

$$\int \vec{F} \cdot d\vec{s} = \int_a^b F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} + F_4 \frac{dw}{dt} \,\mathrm{dt}$$

Then, we can rewrite the derivatives as:

$$\frac{dx}{dt} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}\ ,\quad \frac{dy}{dt} = \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}\ ,$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}\ ,\quad \frac{dw}{dt} = \frac{\partial w}{\partial u}\frac{du}{dt} + \frac{\partial w}{\partial v}\frac{dv}{dt}$$

So, the line integral is rewritten as:

$$\int (F_1 \frac{\partial x}{\partial u} + F_2 \frac{\partial y}{\partial u} + F_3 \frac{\partial z}{\partial u} + F_4 \frac{\partial w}{\partial u}) \frac{du}{dt} + (F_1 \frac{\partial x}{\partial v} + F_2 \frac{\partial y}{\partial v} + F_3 \frac{\partial z}{\partial v} + F_4 \frac{\partial w}{\partial v}) \frac{dv}{dt} \,\mathrm{dt}$$

As in the previous case we use Green's theorem to rewrite this as:

$$\iint \frac{\partial (F_1 \frac{\partial x}{\partial v} + F_2 \frac{\partial y}{\partial v} + F_3 \frac{\partial z}{\partial v} + F_4 \frac{\partial w}{\partial v})}{\partial u} - \frac{\partial (F_1 \frac{\partial x}{\partial u} + F_2 \frac{\partial y}{\partial u} + F_3 \frac{\partial z}{\partial u} + F_4 \frac{\partial w}{\partial u})}{\partial v} dA$$

This simplifies to the expression:

$$\iint \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}-\frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)$$
$$+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)+\left(\frac{\partial F_1}{\partial w}-\frac{\partial F_4}{\partial x}\right)\left(\frac{\partial w}{\partial u}\frac{\partial x}{\partial v}-\frac{\partial w}{\partial v}\frac{\partial x}{\partial u}\right)$$
$$+\left(\frac{\partial F_4}{\partial y}-\frac{\partial F_2}{\partial w}\right)\left(\frac{\partial y}{\partial u}\frac{\partial w}{\partial v}-\frac{\partial y}{\partial v}\frac{\partial w}{\partial u}\right)+\left(\frac{\partial F_3}{\partial w}-\frac{\partial F_4}{\partial z}\right)\left(\frac{\partial z}{\partial u}\frac{\partial w}{\partial v}-\frac{\partial w}{\partial u}\frac{\partial z}{\partial v}\right)dA$$

If we use the notation: $(\nabla \mathbin{⧗} \vec{F})_i$ to represent the i-th column of the “4-curl” matrix, and $(\overrightarrow{T_u} \mathbin{⧗} \overrightarrow{T_v})_i$ to represent the i-th column of that matrix then we can write the identity:

$$\int \vec{F}\cdot d\vec{s} = \iint (\nabla \mathbin{⧗} \vec{F})_1 \cdot \left(\overrightarrow{T_u} \mathbin{⧗} \overrightarrow{T_v}\right)_1 + \left(\nabla \mathbin{⧗} \vec{F}\right)_2 \cdot \left(\overrightarrow{T_u} \mathbin{⧗} \overrightarrow{T_v}\right)_2$$
$$-\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)+\left(\frac{\partial F_3}{\partial w}-\frac{\partial F_4}{\partial z}\right)\left(\frac{\partial z}{\partial u}\frac{\partial w}{\partial v}-\frac{\partial w}{\partial u}\frac{\partial z}{\partial v}\right)dA$$

This can be written for the general “n-curl” to relate line integrals of a vector field $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ . \\ . \\ . \\ F_n \end{bmatrix}$ on the boundaries of surfaces which are maps from $\mathbb{R}^2$ to $\mathbb{R}^n$ that are parametrized as $\Phi(\mathrm{u},\mathrm{v}) = (x_1(\mathrm{u},\mathrm{v}), x_2(\mathrm{u},\mathrm{v}), x_3(\mathrm{u},\mathrm{v}) \dots \dots\ x_n(\mathrm{u},\mathrm{v}))$. The more general identity here is:

$$\int \vec{F}\cdot d\vec{s} = \iint \sum_{i=1}^{n-2} (\nabla \mathbin{⧗_n} \vec{F})_i \cdot \left(\overrightarrow{T_u} \mathbin{⧗_n} \overrightarrow{T_v}\right)_i - (n-3)\left(\frac{\partial F_2}{\partial x_1}-\frac{\partial F_1}{\partial x_2}\right)\left(\frac{\partial x_1}{\partial u}\frac{\partial x_2}{\partial v}-\frac{\partial x_2}{\partial u}\frac{\partial x_1}{\partial v}\right)$$
$$+\sum_{i=3}^{n}\sum_{j>i}^{n}\left(\frac{\partial F_i}{\partial x_j}-\frac{\partial F_j}{\partial x_i}\right)\left(\frac{\partial x_i}{\partial v}\frac{\partial x_j}{\partial u}-\frac{\partial x_j}{\partial v}\frac{\partial x_i}{\partial u}\right)dA$$

Which holds for all $n > 3$. This shows that the line integral about a curve in $\mathbb{R}^n$ which is the boundary of some surface is 0 if this vector field is a gradient field. Because of this, we can conclude that the work done along any path from one point to another is equal if there is a conservative force field, and the work done along a closed loop is zero.

Notes: coordinate system should be modified for vectors in 4-space that have three non-zero entries so that the x and y coordinates contain non-zero values.

The hypercross product of any dimension n, while not useful for curves (1 degree of freedom), 3-manifolds (3 degrees of freedom), or any higher dimensional manifolds, can be useful for evaluating areas of surfaces (geometric objects with 2 degrees of freedom) as well as the integral of scalar functions over these surfaces in any dimension n. For curves, the arc length formula should be used, while for 3 and higher dimensional manifolds wedge products and differential forms should be used to find the content since these generate the volume forms of vectors in $\mathbb{R}^n$. In order to find relate the flow a vector field on the boundary of a manifold to the flux through the manifold differential forms should be used.

The area of a surface in $\mathbb{R}^n$ is given by:

$$A(S) = \iint \frac{\sqrt{\det\left([\vec{a} \mathbin{\bar{\underline{\times}}}_{\mathrm{n}} \vec{b}\,]^T [\vec{a} \mathbin{\bar{\underline{\times}}}_{\mathrm{n}} \vec{b}]\right)}}{\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)^{n-3}} dudv$$

And the integral of a scalar valued function $f(\Phi(\mathrm{u},\mathrm{v}))$ is

$$\iint f dS = \iint f(\Phi(\mathrm{u},\mathrm{v})) \frac{\sqrt{\det\left([\vec{a} \mathbin{\bar{\underline{\times}}}_{\mathrm{n}} \vec{b}\,]^T [\vec{a} \mathbin{\bar{\underline{\times}}}_{\mathrm{n}} \vec{b}]\right)}}{\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)^{n-3}} dudv$$

It should be noted that unlike the wedge product which gives the volume spanned by the vectors whose product is taken, the hypercross product gives the area scaled by a factor that is directly proportional to the dimension of the vectors. Unlike the wedge product, it also cannot be applied to more than two vectors and does not produce a bivector or "2-blade" when the product of two vectors is taken for any dimension except D=4. Another difference is that this produces the orthogonal complement of the two vectors whose product is taken regardless of the dimension.

VI. Inverse n-hypercross product

We can also find the product of a vector and a matrix to find the orthogonal complement of a vector and a bivector using the same method as before. Given a vector $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ and a matrix $B = \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \\ b_4 & c_4 \end{bmatrix}$, the inverse 4-hypercross product would be:

$$\vec{a} \,⧖\, B = \begin{bmatrix} a_1 b_2 (a_4 c_3 - a_3 c_4) + a_1 c_2 (a_3 b_4 - a_4 b_3) + a_1 a_2 (b_3 c_4 - c_3 b_4) \\ (a_1 b_4 - a_4 b_1)(a_1 c_3 - a_3 c_1) - (a_1 b_3 - a_3 b_1)(a_1 c_4 - a_4 c_1) \\ (a_1 b_2 - a_2 b_1)(a_1 c_4 - a_4 c_1) - (a_1 c_2 - a_2 c_1)(a_1 b_4 - a_4 b_1) \\ (a_1 c_3 - a_3 c_1)(a_1 b_2 - a_2 b_1) - (a_1 c_2 - a_2 c_1)(a_1 b_3 - a_3 b_1) \end{bmatrix}$$

The inverse 5-hypercross product as well as the higher dimensional analogues are significantly more complicated.

VII. References and Appendix

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