On the relationship between the internal n-angles of hypercubes and the Riemann zeta function: an investigation

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I. Abstract

An investigation was conducted to study the relationship between the values of the internal n-angles between the facets of hypercubes, polytopes with Schlafli symbol {4,3,3,3….3} (at the corners of these objects), and particular values of the Riemann zeta function at even, positive integers. Calculations were performed for dimensions $4n$ and $4n+1$ for $1 \leq n \leq 6$. No immediately apparent relation was found for any dimension other than those that were powers of two, $d = 4, 8, 16$. The study was therefore furthered to check if the relation that was found for these dimensions held for higher powers of two. It was found that the same rule applied to relating the internal n-angles of the n-hypercube to the Riemann zeta function of all dimensions 2^k for $2 \leq k \leq 8$, thus the relation held for all dimensions that are powers of two up to $d = 256$. No higher dimensions were checked as the numbers being calculated reached the limits of the ability to give meaningful results for the calculator being used. The von-Staudt Clausen theorem and conjectures related to the Fermat numbers were invoked to confirm that this relation held for many dimensions that satisfy the property of being a power of 2. More general formulas that hold for all dimensions that are powers of 2 were derived towards the end.

II. Introduction

Concepts from number theory such as irrational numbers often have important implications in geometry. Irrational numbers that are square roots of natural numbers are related to the diagonals of n-cubes, ratios found in perfect triangles, as well as the characteristics of polytopes in any dimension. This study was motivated by the common relations between these two areas of mathematics, and an attempt to see if there was any connection in terms of angles.

The common 2-dimensional notion of an angle can be generalized to any dimension by recognizing that an angle when measured in radians is related to the proportion of the total normalized circumference 2π that is traversed by an arc of the circle that is between two lines that pass through and meet at the center of the circle. If we look at a sphere and the three planes that meet and pass through the center point, this will give us the 3-angle of an intersection, commonly known as the solid angle and measured in steradians. For the purpose of simplicity, the equivalent for higher dimensions n will be called the n-angle and the measure term will be called n-radians in the remaining portion of this paper.

Since a circle's circumference is 2πr, the maximum angle before repeated equivalent angles will be found is 2π radians, or the circumference divided by the radius. The maximum 3-angle in a sphere would therefore be 4π steradians, the surface area divided by the radius squared. The general maximum n-angle in an n-dimensional space is the total facet content of the ball divided by the radius raised to the power n-1. For example, the maximum 5-angle in a 5-dimensional space would be $\frac{\frac{8\pi^2}{3}r^4}{r^4} = \frac{8\pi^2}{3}$. This relation is used to calculate the n-angles in the corners of hypercubes. The formula for the surface content of a ball of dimension n is

$$\frac{d}{dR}V_n(R) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} R^{n-1}$$

where $V_n(R)$ is the content of the ball of dimension n, $\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} R^n$, and $\Gamma(x)$ is the gamma function, $\int_0^\infty t^{x-1}e^{-t}dt$ which satisfies the property $\Gamma(x+1) = x\Gamma(x)$. Therefore, the regularized facet content or maximum n-angle before repetition is

$$\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

The corners of a hypercube in any dimension therefore sum to equal $\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ n-radians, since the number of n-ants (n-dimensional equivalents of quadrants and octants) in n-dimensions is the same as the number of corners of the n-hypercube, meaning that the full n-angle covered by the n-sphere is equivalent to the sum of the angles of the corners of an n-hypercube. Since there are 2^n n-ants in an n-dimensional space, as well as 2^n corners in an n-cube which have an equal n-angle, the measure of each n-angle in n-radians is $\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})2^n} = \frac{2^{1-n}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$.

The Riemann zeta function is a special function from analysis and number theory that is considered to be important in various areas of mathematics. The Riemann zeta function is related to the distribution of prime numbers and the prime number theorem, arises in definite integration, and is prominent in physics. The value of the function at positive integer even numbers is related to the Basel problem of the 17th and 18th centuries. It is also the subject matter of the Riemann zeta hypothesis problem, which posits that all the non-trivial zeroes of this function in a complex space lie on the critical line $\frac{1}{2} + it$. The function itself is defined in terms of the infinite sum

$$\zeta(\mathrm{n}) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

The values of this function at positive, even integers are known to be related to the mathematical constant π as well as the Bernoulli numbers B_n and the factorial function. They are related by the following formula:

$$\zeta(2\mathrm{k}) = \frac{(-1)^{k+1}(2\pi)^{2k} B_{2k}}{2(2k)!}$$

This yields the first few values of the zeta function that are positive, even integers to be:

$$\zeta(2) = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{\pi^4}{90}$$

$$\zeta(6) = \frac{\pi^6}{945}$$

$$\zeta(8) = \frac{\pi^8}{9450}$$

Upon noticing the similarity of these values of the zeta function with the n-angles at the corners of hypercubes, at dimensions $4n$ and $4n + 1$ for $n \geq 1$, namely the first one where the corner 4-angle of a 4-cube is $\frac{\pi^2}{8}$ and the value of $\zeta(2)$ is $\frac{\pi^2}{6}$, we are prompted to further investigate to see if there is a relationship between this concept of the Riemann zeta function and the geometry of n-hypercubes.

III. Hypercube Internal Angles (Calculation)

The internal n-angles of n-hypercubes of the aforementioned dimensionality in units of n-radians were computed. They were then compared with the zeta function $\zeta(k)$, where $k = \frac{n}{2}$ for dimensions $n = 4y$, and $\frac{n-1}{2}$ for dimensions $n = 4y + 1$. The internal n-angle at the corners, which will be termed for the rest of this paper as $Ang(C_n)$, measured in n-radians, was factorized into the Riemann zeta function of the aforementioned evaluation, and another term which was prime factorized in order to see if there was a pattern that related the Riemann zeta function to the corner. The results are given in the following table.

Dimension $\boldsymbol{n}$	$\boldsymbol{Ang(C_n)}$	$\boldsymbol{Ang(C_n) = x * \zeta(k)}$
4	$\frac{\pi^2}{8}$	$\frac{3}{4} * \frac{\pi^2}{6} = \frac{3}{2^2} * \frac{\pi^2}{6}$
5	$\frac{\pi^2}{12}$	$\frac{1}{2} * \frac{\pi^2}{6} = \frac{1}{2^1} * \frac{\pi^2}{6}$
8	$\frac{\pi^4}{768}$	$\frac{15}{128} * \frac{\pi^4}{90} = \frac{5 * 3}{2^7} * \frac{\pi^4}{90}$
9	$\frac{\pi^4}{1680}$	$\frac{3}{56} * \frac{\pi^4}{90} = \frac{3}{2^3 * 7} * \frac{\pi^4}{90}$
12	$\frac{\pi^6}{245670}$	$\frac{35}{16384} * \frac{\pi^6}{945} = \frac{7 * 5}{2^{14}} * \frac{\pi^6}{945}$
13	$\frac{\pi^6}{1441440}$	$\frac{3}{4576} * \frac{\pi^6}{945} = \frac{3}{2^5 * 11 * 13} * \frac{\pi^6}{945}$
16	$\frac{\pi^8}{165150720}$	$\frac{15}{262144} * \frac{\pi^8}{9450} = \frac{5 * 3}{2^{18}} * \frac{\pi^8}{9450}$
17	$\frac{\pi^8}{518918400}$	$\frac{1}{549120} * \frac{\pi^8}{9450} = \frac{1}{2^8 * 5 * 11 * 13} * \frac{\pi^8}{9450}$
20	$\frac{\pi^{10}}{190253629440}$	$\frac{33}{67108864} * \frac{\pi^{10}}{93555} = \frac{3 * 11}{2^{26}} * \frac{\pi^{10}}{93555}$
21	$\frac{\pi^{10}}{670442572800}$	$\frac{3}{21498880} * \frac{\pi^{10}}{93555} = \frac{3}{2^{10} * 5 * 13 * 17 * 19} * \frac{\pi^{10}}{93555}$

24	$\frac{\pi^{12}}{334846387814400}$	$\frac{4095}{2147483648 * 691} * \frac{691\pi^{12}}{638512875} = \frac{13 * 5 * 3 * 21}{2^{31} * 691} * \frac{691\pi^{12}}{638512875}$
25	$\frac{\pi^{12}}{2590590101299200}$	$\frac{15}{60858368 * 691} * \frac{691\pi^{12}}{638512875} = \frac{5 * 3}{2^{13} * 17 * 19 * 21 * 691} * \frac{691\pi^{12}}{638512875}$

Immediate investigation of the table reveals that there is a similarity for all the even dimensions listed, because the denominator of the relating term (the term that is the quotient of $Ang(C_n)$ and ζ(k)), is always a power of 2, excluding the case of $d = 24$. This can be explained however by the fact that the Riemann zeta term in this dimension has a numerator that is not $1 * \pi^{\frac{n}{2}}$ and is instead $691 * \pi^{\frac{n}{2}}$. When factoring out 691 the denominator remains a power of 2. In all the odd dimensions except $d = 5$ the denominator has other prime factors that are not 2. There is no apparent pattern in the numerator of the relating term for odd or even dimensions.

There is no apparent pattern in the total set of terms we have observed therefore in this factorization except for some sort of pattern in the subset of terms that are the even dimensions, $4n$. To look more closely, we can observe what number 2 is exponentiated to in each term.

Dimension	**Exponent of 2 in denominator**
4	2
8	7
12	14
16	18
20	26
24	31

There is no common difference between the exponents, of all orders. There is also no multiplicative factor that relates one term to the next.

If we take an even smaller subset of these dimensions and observe only the dimensions that are powers of 2 (by definition this is a subset of the even dimensions), $d = 4{,}8{,}16$, there is an observable pattern, however. The exponent for $d = 4$ is 2, the exponent for $d = 8$ is 7, and the exponent for $d = 16$ is 18. If we label $n = \log_2 d$, and $m = n - 1$, the sequence 2, 7, 18 is defined by the recursive formula

$$\mu_{m+1} = 2\mu_m + (m + 2)$$

where $\mu_0 = 0$. The next term in the sequence would be 41.

For consistency, we check if there is a pattern to be observed in the numerators of the relating term for these dimensions. The numerator for $d = 4$ is $3 = 3 * 1$, for $d = 8$ it is $15 = 5 * 3$, and for $d = 16$ it is again $15 = 5 * 3$. For the first two cases it is the case that the term is equal to $(\frac{d}{2} + 1)(\frac{d}{2} - 1)$ but we can't use this formula since it is not the case for $d = 16$. We observe however that $(\frac{d}{2} + 1)$ is prime for $d = 4, 8$, but not so for $d = 16$. This notable difference is enough to make this relating term distinguishable, and to prompt further investigation to see if the relating term follows the same rule for dimensions that are higher powers of 2.

IV. Deriving relationship with Riemann Zeta Function from direct calculation

Investigating the relationship between the Riemann Zeta function and the internal angles of a hypercube for $d = 32$ yields the interesting result:

Dimension n	32
$\boldsymbol{Ang(C_n)}$	$\frac{\pi^{16}}{28082093221887344640000}$
$\boldsymbol{Ang(C_n) = x * \zeta(k)}$	$\frac{255}{2^{41} * 3617} * \frac{3617\pi^{16}}{325641566250} = \frac{17 * 15}{2^{41} * 3617} * \frac{3617\pi^{16}}{325641566250}$

The aforementioned pattern is observed for the numerator of the relating term; however, the denominator is slightly different. While 41 is the expected power that 2 is raised to based on the formula we identified for the earlier terms, 3617 is not a power of 2. However, we notice that any value $\zeta(2n)$ can be written in the form

$$\zeta(2n) = \frac{b_n}{a_n}\pi^{2n}$$

where b_n and a_n are coprime, positive integers for all n given in the following table provided by the OEIS:

n	***a$_n$***	***b$_n$***
1	6	1
2	90	1
3	945	1
4	9450	1
5	93555	1
6	638512875	691
7	18243225	2
8	325641566250	3617
9	38979295480125	43867
10	1531329465290625	174611
11	13447856940643125	155366
12	201919571963756521875	236364091
13	11094481976030578125	1315862
14	564653660170076273671875	6785560294
15	5660878804669082674070015625	6892673020804
16	62490220571022341207266406250	7709321041217

We previously saw $b_6 = 691$ for $d = 24$, and now we see $b_8 = 3617$. We can apply the same rule we used for initially distinguishing the denominators as powers of two, and account for this term. Doing this yields a rule that holds for all dimensions that are powers of 2 that we have checked so far.

$$Ang(C_{2^n}) = \frac{(\lambda_{prime} + 1) * (\lambda_{prime} - 1)}{2^{\mu_m} * (b_{2^{n-2}})} * \zeta(2^{n-1})$$

where

$$\mu_{m+1} = 2\mu_m + (m+2)$$

$$\mu_0 = 0$$

$$m = n - 1$$

as explained previously, and λ_{prime} is the first power of 2 smaller than 2^n in which $(\lambda_{prime} + 1)$ is a prime number.

Checking this result for the next few cases requires calculations that include extremely large numbers. We will therefore stop at $n = 8$. The next terms in the sequences would be 88, 183, and 374. The relation was found to hold for all n up to and including 8. The results are in the following tables (sorry for the miniscule text size):

Dimension 2^n	$\boldsymbol{Ang(C_{2^n})}$
64	$\frac{\pi^{32}}{7584230010651321144025484834203462841300680704000000}$
128	$\frac{\pi^{64}}{337323325121478683780454353294367996350443621952124657714538832354016429985548488767462936789327475580873788620800000000000000}$
256	$\frac{\pi^{128}}{174421098849471985214706628571555850350629281352330166168199805026150893824849921025127145496959526882560151456594386881500822704840042883298820595876830634263656036093891189560905695297445442530483286885667431609287462045754096109968645416038117459359359303680000000000000000000000000000}$

Dimension 2^n	$\frac{(\lambda_{prime}+1)*(\lambda_{prime}-1)}{2^{\mu_m}*(b_{2^{n-2}})} * \zeta(2^{n-1})$
64	$\frac{17*15}{2^{88}*7709321041217} * \frac{7709321041217\pi^{32}}{62490220571022341207266406250}$
128	$\frac{17*15}{2^{183}*10678383014786652988638544497914264794 2017} * \frac{106783830147866529886385444979142647942017\pi^{64}}{70161254643337808194150291650798560032775321033675849947561411743164 06250}$
256	$\frac{17*15}{2^{374}*267754707742548082886954405585282394779291459592551740629978686063357792734863530145362663093519862048495908453718017} * \frac{267754707742548082886954405585282394779291459592551740629978686063357792734863530145362663093519862048495908453718017\pi^{128}}{1155901481375376048718061598661841716282623784987101067397115316331066531792461702094959418380053495874174062995252037446754411181755833166510460555313553380453959107398986816406250}$

If the relation holds for n=9 the next term should be $\frac{257*255}{2^{757}*(b_{2^7})} * \zeta(256)$, since 257 is a prime number. Thus, we have found a sufficient formula relating the Riemann zeta function to the internal n-angles of hypercubes, measured in n-radians, for dimensions that are powers of 2 excluding 2 itself, that holds for all cases that have been tested. To test higher cases, say to n=50, would require an immense amount of time and probably a supercomputer. And even if this was performed, there is a limit to what can be tested with a computer, so to see if the formula holds for all n we should start to do some analysis.

V. Direct algebraic derivation of formula from Euler's formula for zeta at positive even integers and a result related to the von Staudt Clausen theorem

We can start by noticing the similarities between the right- and left-hand sides of the equation. The facet content formula includes the gamma function, Γ(x) evaluated at $x = 2^{n-1}$. We note that the Riemann zeta function can be written in terms of the expression

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1} du$$

and that in our equation it is evaluated at $x = 2^{n-1}$. So, we can cancel the gamma function on both sides to get the expression,

$$\frac{(ƛ_{prime} + 1) * (ƛ_{prime} - 1)}{2^{\mu_m} * (b_{2^{n-2}})} * \int_0^\infty \frac{u^{2^{n-1}-1}}{e^u - 1} du = 2^{1-2^n} \pi^{2^{n-1}}$$

For n=2-8. The integral on the left-hand side is non-elementary, and the recursive formula, the $b_{2^{n-2}}$ numbers, as well as the rule for $ƛ_{prime}$ makes it a time-consuming project to try and see what happens in the general case for any n, so this topic will not be discussed in this paper.

Another approach is to calculate the subtended angle at the corners by using the formula that was provided earlier for the Riemann zeta function at even positive integers, which is known as Euler's formula. The formula is presented earlier in this work. Manipulation of the formula solving for π yields

$$\pi^{2k} = \frac{\zeta(2k) * 2(2k)!}{2^{2k} * (-1)^{k+1} * B_{2k}}$$

We set $2k = p$ to get

$$\pi^p = \frac{\zeta(p) * 2(p)!}{2^p * (-1)^{\frac{p}{2}+1} * B_p}$$

The formula for the internal d-angle of a hypercube is $\frac{\pi^{\frac{d}{2}}}{2^{d-1}\Gamma(\frac{d}{2})}$. Setting $d = 2p$ we get

$$\frac{\pi^p}{2^{2p-1}\Gamma(p)}$$

As the formula, which means that

$$Ang(C_{2p})2^{2p-1}\Gamma(p) = \pi^p$$

So we derive a formula for even dimensions

$$Ang(C_{2p}) = \frac{\zeta(\mathrm{p}) * 2(p)!}{2^p * (-1)^{\frac{p}{2}+1} * B_p * 2^{2p-1}\Gamma(p)}$$

Which can be simplified to

$$Ang(C_{2p}) = \frac{\zeta(\mathrm{p}) * p}{(-1)^{\frac{p}{2}+1} * B_p * 2^{3p-2}}$$

Noting that $\Gamma(p) = (p-1)!$, yielding a general formula for even dimensions. The Bernoulli number can be separated into a numerator and a denominator, so we can write the formula:

$$Ang(C_{2p}) = \frac{\zeta(\mathrm{p}) * p * D_p}{(-1)^{\frac{p}{2}+1} * N_p * 2^{3p-2}}$$

Where D_p and N_p are the denominators and numerators respectively. Setting $p = 2^n$ to check if the relationship found previously holds, we find that

$$Ang(C_{2^{n+1}}) = \frac{\zeta(2^n) * 2^n * D_{2^n}}{|N_{2^n}| * 2^{3(2^n)-2}} = \frac{\zeta(2^n) * D_{2^n}}{|N_{2^n}| * 2^{3(2^n)-2-n}}$$

Setting $m = n + 1$ we get

$$Ang(C_{2^m}) = \frac{\zeta(2^{m-1}) * D_{2^{m-1}}}{|N_{2^{m-1}}| * 2^{3(2^{m-1})-1-m}}$$

The denominators of the even Bernoulli numbers are determined from a theorem due to Graham et al (1994) closely related to the von Staudt Clausen theorem which shows that the denominator of the Bernoulli numbers of even indices are derived from the following formula:

$$denom(B_{2k}) = \prod_{(t-1)|2k} t$$

Where t is a prime number and the notation $x|y$ means that y is divisible by x. Since the dimensions are all of the form 2^n, this implies that all t are Fermat primes, in addition to the number 2 (since the only numbers that are of the form $2^n + 1$ that are prime are the Fermat primes). The 2 makes us rewrite the formula as

$$Ang(C_{2^m}) = \frac{\zeta(2^{m-1}) * \prod_{\lambda} F_{\lambda}}{|N_{2^{m-1}}| * 2^{3(2^{m-1})-2-m}}$$

Where $\prod_{\lambda} F_{\lambda}$ denotes the product of the Fermat primes smaller than 2^m.The sequence

$3(2^{m-1}) - 2 - m$ starting at m=2 can be shown algebraically to be equivalent to the sequence we identified previously as

$$\mu_{v+1} = 2\mu_v + (v + 2)$$

$$\mu_0 = 0$$

$v = m - 1$ starting at m=2, since $f(x) = 3(2^{x-1}) - 2 - x$ satisfies the equation

$$f(x) = 2 * f(x - 1) + ((x - 1) + 2)$$

So, to check for which dimensions the relation found previously is confirmed to hold up to, since all the other terms are the same, we therefore only have to refer to the Fermat numbers and conjectures related to their property of being prime. Fermat numbers are all numbers where $n \geq 1$ of the form $2^{2^n} + 1$. The first five Fermat numbers are known to be prime however it is not known whether or not there are any more prime Fermat numbers. This reveals an interesting connection between the Bernoulli numbers and the λ_{prime} term, which is closely related to the von Staudt Clausen theorem, as well as confirmation that the sequence we obtained for μ_m was correct.

So, we find that the rule for λ_{prime} holds for several relatively "low" dimensions but changes in accordance with the change of the nature of the Fermat numbers. We change our criterion for λ_{prime} to be the product of the prime Fermat numbers. It is known that the expression $2^n + 1$ is composite for all integers n that are not Fermat primes. It is conjectured that the first 5 prime Fermat numbers are the only ones that exist. If this conjecture holds, the formula we derived from direct calculation holds for all dimensions, if not, it must be adjusted slightly. It changes from $(\lambda_{prime} + 1) * (\lambda_{prime} - 1)$ to $\prod_{\lambda} F_{\lambda}$ where F_{λ} are all the prime Fermat numbers. However, this change holds for both cases, so this can be termed as the general formula that is confirmed for all dimensions. So, whether or not the first formula is a general formula is directly related to the open question of whether there are infinitely many, more, or only the 5 known Fermat primes. The general formula therefore is

$$Ang(C_{2^n}) = \frac{\prod_{\lambda} F_{\lambda}}{2^{\mu_m} * (b_{2^{n-2}})} * \zeta(2^{n-1})$$

VI. Extension to other polytopes

The hypercube is one of three regular polytopes that exist in any dimension greater than 2, the others being the orthoplex, a polytope with Schlafli symbol {3,3,3….4} and the simplex, {333….3}. The orthoplex, also known as the cross polytope, is the dual of the hypercube. Future study may see if there is a relationship between the internal n-angles of other polytopes, regular and nonregular, and the Riemann zeta function.

VII. Hypercube zeta conjecture – an extension of conjectures related to the Fermat primes

From knowledge of the Bernoulli numbers, the direct calculation which has shown that the relation holds exactly for, $n = 2,3,\ldots,8$, and synthesizing this with the algebraic formulas we conclude that the relationship between the internal 2^n-angles between the 2^n-1 facets of hypercubes and the Riemann zeta function in which it is proportional to the ratio $\frac{(\lambda_{prime}+1)*(\lambda_{prime}-1)}{2^{\mu_m}*(b_{2^n-2})}$, with all the aforementioned definitions ascribed (the second criterion for all dimensions, the first one for several "low" dimensions) , holds for dimensions that have dimensionality 2^n with $n \geq 2$. If the conjecture that there are only 5 Fermat primes holds then it would be proven that this formula holds for all dimensions. If not, we can use the two formulas given in this paper to describe the relationship between the Riemann zeta function and the hypercube's internal n-angles. They are related by a term λ_{prime} which is itself a power of 2 with a property that was described in the previous section, μ_m, another term that 2 is raised to the power of and that is given by the sequences listed in the preceding sections, as well as b_{n}, which is a term that is found in the numerator of the zeta function at positive, even, integer values. Information about prime numbers is in most of the terms on the right hand side, so the relating term which relates these quantities related to prime numbers and n-cubes is the μ_m sequence, which has been proven to be the case for a few select cases as well as for the general case from the formula relating to Bernoulli numbers. Therefore, this result is a significant sequence since it relates to the internal n-angles of n-cubes and the zeta function at powers of 2. This sequence shows up in other places in mathematics such as the hook partitions of combinatorics and number theory. So, all in all, it seems as if Bernoulli numbers, zeta functions, Fermat primes, hook partitions, and n-cubes have a strange relationship with one another. This suggests a relationship between number theory and geometry which may be extendable to other regular and uniform polytopes.

VIII. Extension to the Dirichlet Lambda function

Using a known relationship between the Riemann zeta function and the Dirichlet lambda function, we can extend this result to relate the internal k-angles of hypercubes to that function as well. The Dirichlet lambda function is defined as such:

$$\lambda(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^x}$$

It can be related to the Riemann zeta function by the following formula:

$$\lambda(v) = \frac{2^v - 1}{2^v} \zeta(v)$$

By substituting this relationship into the previous equation, we get the relationship between the internal k-angles of hypercubes in dimensions that are powers of 2 to the Dirichlet lambda function.

$$Ang(C_{2^n}) = \frac{\prod_{\lambda} F_{\lambda}}{b_{2^{n-2}}} * \lambda(2^{n-1}) * \frac{2^{2^{n-1}}}{2^{3(2^{n-1})-2-n} * (2^{2^{n-1}} - 1)}$$

This can be simplified to the following formula:

$$Ang(C_{2^n}) = \frac{\prod_{\lambda} F_{\lambda} * \lambda(2^{n-1})}{b_{2^{n-2}} * 2^{2^n-2-n} * (2^{2^{n-1}} - 1)}$$

The expression $2^n - 2 - n$ is the generating expression for the sequence 0, 3, 10, 25, 56, etc. The other term is a subset of the Mersenne numbers, as it is equal to $M_{2^{n-1}}$ for dimension 2^n.

IX. References and Appendix

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Calculations performed using Desmos Scientific, Number Empire, and Big Number Calculator:

https://www.calculator.net/big-number-calculator.html

https://www.desmos.com/scientific

https://www.numberempire.com/

Calculator Screenshots for equivalencies between the two terms:

$$\frac{\pi^2}{8} = 1.23370055$$

$$\frac{3 \cdot 1}{2^2 \cdot 1} \cdot \frac{\pi^2}{6} = 1.23370055$$

$$5 \cdot 3 \cdot \frac{\pi^4}{2^7 \cdot 1 \cdot 90} = 0.126834754$$

$$\frac{\pi^4}{768} = 0.126834754$$

$$\frac{\pi^{16}}{2\,808\,209\,322\,188\,738\,864\,000} = 3.20603668 \times 10^{-14}$$

$$\frac{(17 \cdot 15)}{3\,617 \cdot 2^{41}} \cdot \frac{3\,617\pi^{16}}{325\,641\,566\,250} = 3.20603668 \times 10^{-14}$$

$$5 \cdot 3 \cdot \frac{\pi^8}{2^{18} \cdot 1 \cdot 9\,450} = 5.74537672 \times 10^{-5}$$

$$\frac{\pi^8}{165\,150\,720} = 5.74537672 \times 10^{-5}$$

$$\frac{\left(17 \cdot 15 \cdot \pi^{32}\right)}{2^{88} \cdot 62\,490\,220\,571\,022\,341\,207\,266\,406\,250} = 1.06877043 \times 10^{-37}$$

$$\frac{\pi^{32}}{75\,842\,300\,106\,513\,211\,440\,254\,848\,342\,034\,628\,413\,0} = 1.06877043 \times 10^{-37}$$

$$\frac{(17 \cdot 15)}{2^{183}} \cdot \frac{}{7\,016\,125\,464\,333\,780\,819\,415\,029\,165\,0} = 1.94780502 \times 10^{-94}$$

$$\frac{\pi^{64}}{53\,294\,367\,996\,350\,443\,621\,952\,124\,657\,714\,538\,832\,354} = 1.94780502 \times 10^{-94}$$

$$\frac{(17 \cdot 15)}{2^{374}} \cdot \frac{}{1\,155\,901\,481\,375\,376\,048\,718\,061\,598} = 2.47505405 \times 10^{-227}$$

$$\frac{\pi^{128}}{56\,594\,386\,881\,500\,822\,704\,840\,042\,883\,298\,820\,595\,87} = 2.47505405 \times 10^{-227}$$